MATH 590: QUIZ 10 SOLUTIONS

Name:

Find the singular value decomposition of $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ by following the steps below. You may use the fact that $p_{A^tA}(x) = x(x-2)(x-3)$. **Be sure to label each step in your solution.** Each step is worth 2 points.

1. Calculate A^tA.
2. Find the non-zero eigenvalues of A^tA: $\lambda_1 > \lambda_2 > 0$.
3. Find: (i) A unit eigenvector u_1 of λ_1, a unit eigenvector u_2 for λ_2 and a unit vector u_3 such that u_1, u_2, u_3 is an orthonormal basis for $\mathbb{R}^3$.
4. Set $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, $v_1 = \frac{1}{\sigma_1}Au_1$, and $v_2 = \frac{1}{\sigma_2}Au_2$. Show that v_1, v_2 is an orthonormal basis for $\mathbb{R}^2$.
5. Let P be the orthogonal matrix whose columns are u_1, u_2, u_3, Q the orthogonal matrix whose columns are v_1, v_2, and $\sum = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix}$. Verify that $A = Q\sum P^t$.

Solution. For 1: $A^tA = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

For 2: The non-zero eigenvalues of A^tA are: $3 > 2$.

For 3: E_3 is the null space of $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, so we can take $u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

E_2 is the nullspace of $\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so we can take $u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. By inspection, if we take $u_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, we have an orthonormal basis for $\mathbb{R}^3$.

For 4: Direct calculation gives, $\sigma_1 = \sqrt{3}, \sigma_2 = \sqrt{2}, v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is clearly an orthonormal basis for $\mathbb{R}^2$.

For 5: We have $P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sum = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$. Therefore,

$$Q\sum P^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{3} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} = A$$